Invariantly Characterize Numerical Spacetimes

Abstract: This topic will cover the fundamental concepts related to the invariants characterizing numerical spacetimes. Initially, I intend to elucidate the methodology employed in computing the electric and magnetic components of the Weyl tensor through the utilization of the 3+1 slicing formulation. And the presentation will include an exploration of the relationship between the Weyl scalar and the electric and magnetic components in terms of null tetrad basis. Finally, the Petrov classification and fundamental invariants are presented.

Motivation

- In recent years, it has become increasingly common in numerical relativity to extract gravitational wave information in terms of the components of the Weyl curvature tensor with respect to a frame of null vectors, using what is known as the Newman–Penrose formalism.

- They can be used to compare a numerical solution to an exact solution, or two numerical solutions corresponding to the same physical system but computed in different gauges.

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3+1 slicing

Consider the 3+1 formalism:

This foliates the spacetime into spatial hypersurfaces with a **spatial metric** $\gamma_{\alpha\beta}$ and a normal unit timelike vector n^α: $n^\alpha n_\alpha = -1$ and $n^\alpha \gamma_{\alpha\beta} = 0$.

The time coordinate is chosen such that it is constant on each of the spatial slices, covered by the space coordinates, thereby defining *coordinates adapted to the foliation*. Each time slice is mapped to the next by the **lapse function** α and the **shift vector** β^μ.

In this foliation-adapted coordinates, we can choose $n_\mu = (-\alpha, 0,0,0)$ and $n^\mu = \left(1/\alpha, -\beta^i/\alpha\right)$, $\beta^\mu = \left(0, \beta^i\right)$ and $\beta_\mu = \left(\beta_k \beta^k, \beta_j\right)$ with $\beta_i = \gamma_{ij}\beta^j$. So, the spacetime metric $g_{\mu\nu}$ and spatial metric $\gamma_{\mu\nu}$:

$$\gamma_{\mu\nu} = g_{\mu\nu} + n_\mu n_\nu = \begin{pmatrix} 0 & 0 \\ 0 & g_{ij} \end{pmatrix} \tag{1}$$

$$g^{\mu\nu} = \gamma^{\mu\nu} - n^\mu n^\nu = \begin{pmatrix} -\alpha^{-2} & \alpha^{-2}\beta^i \\ \alpha^{-2}\beta^i & \gamma^{ij} - \alpha^{-2}\beta^i\beta^j \end{pmatrix} \tag{2}$$

$$ds^2 = -\alpha^2 dt^2 + \gamma_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt) \tag{3}$$

3+1 slicing

The evolution of $\gamma_{\alpha\beta}$ is given by the extrinsic curvature, where $\mathcal{L}_{\boldsymbol{n}}$ represents the Lie derivative along n^{μ}. We can write

$$K_{\alpha\beta} = -\frac{1}{2}\mathcal{L}_{\mathbf{n}}\gamma_{\alpha\beta} \\ = -\nabla_{\alpha}n_{\beta} - n_{\alpha}n^{\mu}\nabla_{\mu}n_{\beta} \quad (4)$$

orthogonal to n^{α}: $n^{\alpha}K_{\alpha\beta} = 0$

Observably, the spatial metric $\gamma_{\alpha\beta}$ and extrinsic curvature $K_{\alpha\beta}$ cannot be chosen arbitrarily. Instead, they have to satisfy certain equations.

$$\gamma_a^p\gamma_b^q\gamma_c^r\gamma_d^s\,{}^{(4)}R_{pqrs} = R_{abcd} + K_{ac}K_{bd} - K_{cb}K_{ad} \quad (5)$$

$$\gamma_a^p\gamma_b^q\gamma_c^r n^s\,{}^{(4)}R_{pqrs} = -D_aK_{bc} + D_bK_{ac} \quad (6)$$

$$n^c n^d\gamma_a^q\gamma_b^r\,{}^{(4)}R_{drcq} = -\mathcal{L}_{\mathbf{n}}K_{ab} - \frac{1}{\alpha}D_aD_b\alpha - K_{ca}K_b^c \quad (7)$$

3+1 slicing

So, we can get the equivalent form of Einstein's equations: coupled evolution equations and the constraint equations:

$$16\pi\rho = R + K^2 - K_{ij}K^{ij} \tag{8}$$

$$8\pi S^i = D_j(K^{ij} - \gamma^{ij}K) \tag{9}$$

constraint equations

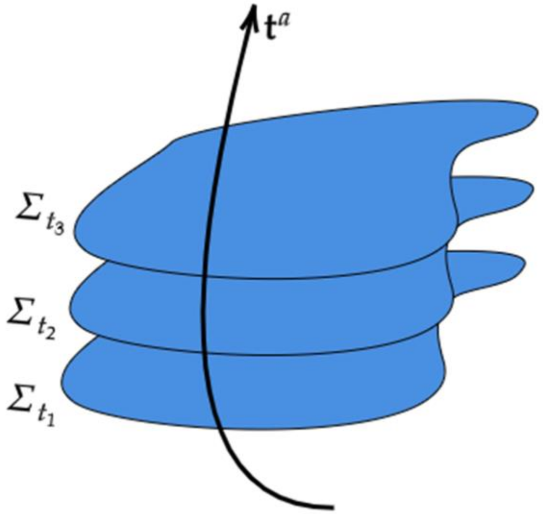
$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i\beta_j + D_j\beta_i \tag{10}$$

$$\partial_t K_{ij} = -D_iD_j\alpha + \alpha(R_{ij} - 2K_{ik}K^k_j + KK_{ij}) - 8\pi\alpha(S_{ij} - \frac{1}{2}\gamma_{ij}(S-\rho)) + \beta^k D_k K_{ij} + K_{ik}D_j\beta^k + K_{kj}D_i\beta^k \tag{11}$$

evolution equations

where

$$\rho \equiv n_a n_b T^{ab}, S^i = -\gamma^{ij}n^a T_{aj}, S_{ij} = \gamma_{ia}\gamma_{jb}T^{ab}, S = \gamma^{ij}S_{ij}$$



Weyl tensor

The Riemann tensor has more independent components than the Ricci tensor, which implies that it can be decomposed in terms of the Ricci and an additional object known as the **Weyl tensor**, and defined for an n-dimensional spacetime as

$$C_{\alpha\beta\mu\nu} := R_{\alpha\beta\mu\nu} - \frac{2}{n-2}\left[g_{\alpha[\mu}R_{\nu]\beta} - g_{\beta[\mu}R_{\nu]\alpha}\right] + \frac{2}{(n-1)(n-2)}g_{\alpha[\mu}g_{\nu]\beta}R. \tag{12}$$

Moreover, we can also easily see that it is traceless,

$$C^{\alpha}{}_{\mu\alpha\nu} = 0 \tag{13}$$

Given an arbitrary timelike unit vector n^{μ}, we define the electric $E_{\mu\nu}$ and magnetic $B_{\mu\nu}$ parts of the Weyl tensor as:

$$E_{\mu\nu} := n^{\alpha}n^{\beta}C_{\alpha\mu\beta\nu} \tag{14}$$

$$B_{\mu\nu} := n^{\alpha}n^{\beta}C^{*}_{\alpha\mu\beta\nu} \tag{15}$$

where $C^{*}_{\alpha\mu\beta\nu}$ is the so-called dual Weyl tensor:

$$C^{*}_{\alpha\mu\beta\nu} := \frac{1}{2}C_{\alpha\beta\lambda\sigma}\varepsilon^{\lambda\sigma}{}_{\mu\nu} \tag{16}$$

Weyl tensor

If we now take the vector n^μ to be the unit normal vector to the spacelike hypersurfaces in the 3+1 formalism, we can write the electric and magnetic tensors in 3+1 formalism as

$$E_{ij} = R_{ij} + KK_{ij} - K_{im}K^m{}_j - 4\pi\left[S_{ij} + \frac{\gamma_{ij}}{3}(4\rho - S)\right] \tag{17}$$

$$B_{ij} = \epsilon_i{}^{mn}\left[D_m K_{nj} - 4\pi\gamma_{jm}j_n\right] \tag{18}$$

where D_m is 3-dimensional covariant derivative.

Further more, E_{ij} and B_{ij} are symmetry and traceless,

$$\begin{aligned} \epsilon^{ija}B_{ij} &= \epsilon^{ija}\epsilon_{imn}\left(D^m K^n{}_j - 4\pi\delta^m_j j^n\right) \\ &= \left(\delta^j_m\delta^a_n - \delta^j_n\delta^a_m\right)\left(D^m K^n{}_j - 4\pi\delta^m_j j^n\right) \\ &= \left(D^m K^a{}_m - 4\pi\delta^m_m j^a\right) - \left(D^a K^n{}_n - 4\pi\delta^a_n j^n\right) \\ &= D^m K^a{}_m - 8\pi j^a - D^a K = 0 \end{aligned} \tag{19}$$

And the Weyl tensor can be decomposed as,

$$C_{\alpha\beta\mu\nu} = 2\left(l_{\alpha[\mu}E_{\nu]\beta} - l_{\beta[\mu}E_{\nu]\alpha} - n_{[\mu}B_{\nu]\lambda}\epsilon^\lambda{}_{\alpha\beta} - n_{[\alpha}B_{\beta]\lambda}\epsilon^\lambda{}_{\mu\nu}\right) \tag{20}$$

where $l_{\mu\nu} \coloneqq g_{\mu\nu} + 2n_\mu n_\nu$.

The tetrad formalism

Up to this point we have assumed that the components of tensors are always expressed in terms of a coordinate basis $\{\vec{e}_\mu\}$. However, in many cases it is particularly useful to work instead with a basis that is independent of the coordinates.

At every point of spacetime, consider a set of four linearly independent vectors $\{\vec{e}_{(a)}\}$, such that,

$$\vec{e}_{(a)}\vec{e}_{(b)} = \eta_{(a)(b)} \tag{21}$$

with $\eta_{(a)(b)}$ is a constant matrix independent of the position in spacetime. In such a case the set of vectors $\{\vec{e}_{(a)}\}$ is called a **tetrad**.

So the tensor components can be expressed in the tetrad basis. For example, for a rank 2 tensor we have

$$T_{(a)(b)} = T_{\mu\nu} e_{(a)}{}^{\mu} e_{(b)}{}^{\nu}, \qquad T_{\mu\nu} = T_{(a)(b)} e^{\mu}{}_{(a)} e^{\nu}{}_{(b)} \tag{22}$$

Basically, covariant derivatives can be express in terms of the tetrad basis as

$$A_{(a)|(b)} := e_{(a)}{}^{\mu} e_{(b)}{}^{\nu} \nabla_{\nu} A_{\mu} = A_{(a),(b)} - \gamma^{(c)}{}_{(a)(b)} A_{(c)} \tag{23}$$

where

$$A_{(a),(b)} := e_{(b)}{}^{\mu} \partial_{\mu} A_{(a)} \quad \gamma_{(a)(b)(c)} := e_{(a)}{}^{\mu} e_{(c)}{}^{\nu} \nabla_{\nu} e_{(b)\mu} \tag{24}$$

We then see that the $\gamma^{(c)}{}_{(a)(b)}$ are nothing more than the connection coefficients in the tetrad basis.

$$\gamma_{(a)(b)(c)} = e_{(a)}{}^{\mu} e_{(c)}{}^{\nu} \nabla_{\nu} e_{(b)\mu}$$

$$\gamma_{(a)(b)(c)} e^{(a)}{}_{\mu} = e_{(c)}{}^{\nu} \nabla_{\nu} e_{(b)\mu}$$

$$\gamma^{(a)}{}_{(b)(c)} \vec{e}_{(a)} = e_{(c)}{}^{\nu} \nabla_{\nu} \vec{e}_{(b)}$$

According to the equation(21)

$$\nabla_{\mu}(\eta_{(a)(b)}) = \nabla_{\mu}(e_{(a)}{}^{\nu} e_{(b)\nu}) = 0 \Rightarrow e_{(a)}{}^{\nu} \nabla_{\mu} e_{(b)\nu} = -e_{(b)\nu} \nabla_{\mu} e_{(a)}{}^{\nu} \tag{25}$$

Obviously,

$$\gamma_{(a)(b)(c)} = -\gamma_{(b)(a)(c)} \tag{26}$$

The tetrad formalism

Advantages of tetrad:

- the tetrad components of any geometric object behave as scalars with respect to coordinate changes.

- This means that in a general four-dimensional spacetime there are only 24 independent $\gamma_{(a)(b)(c)}$, which is in contrast with the 40 independent components of the Christoffel symbols $\Gamma^{\alpha}_{\mu\nu}$in a coordinate basis.

The basic idea of the Newman–Penrose formalism is to introduce a tetrad of **null vectors**.

Typically, we choose the vector $e^{\mu}_{(0)}$ as the unit normal to the spatial hypersurfaces $e^{\mu}_{(0)} = n^{\mu}$, $e^{\mu}_{(1)}$ as the unit radial vector in spherical coordinates, and $\left\{e^{\mu}_{(2)}, e^{\mu}_{(3)}\right\}$ as unit vectors in the angular directions. Once we have an orthonormal basis, we can construct the four null vectors:

Real Vectors

$$l^{\mu} := \frac{1}{\sqrt{2}}\left(e^{\mu}_{(0)} + e^{\mu}_{(1)}\right) \tag{27}$$

$$k^{\mu} := \frac{1}{\sqrt{2}}\left(e^{\mu}_{(0)} - e^{\mu}_{(1)}\right) \tag{28}$$

Complex Vectors

$$m^{\mu} := \frac{1}{\sqrt{2}}\left(e^{\mu}_{(2)} + ie^{\mu}_{(3)}\right) \tag{29}$$

$$\bar{m}^{\mu} := \frac{1}{\sqrt{2}}\left(e^{\mu}_{(2)} - ie^{\mu}_{(3)}\right) \tag{30}$$

what is known as a **null tetrad**, such as

$$\begin{aligned} l_{\mu}l^{\mu} = k_{\mu}k^{\mu} = m_{\mu}m^{\mu} = \bar{m}_{\mu}\bar{m}^{\mu} = 0, \\ l_{\mu}m^{\mu} = l_{\mu}\bar{m}^{\mu} = k_{\mu}m^{\mu} = k_{\mu}\bar{m}^{\mu} = 0, \\ l_{\mu}k^{\mu} = -m_{\mu}\bar{m}^{\mu} = -1 \end{aligned} \tag{31}$$

The Newman–Penrose tetrad

The definition of the null tetrad $\{l^\mu, k^\mu, m^\mu, \bar{m}^\mu\}$ is based on the choice of the original orthonormal tetrad $\{\vec{e}_{(a)}\}$. Such transformations are usually separated into three distinct classes:

- Null rotations of class I which leave the vector l^μ unchanged[1]:

$$l^\mu \to l^\mu, \qquad m^\mu \to m^\mu + a l^\mu$$
$$k^\mu \to k^\mu + \bar{a} m^\mu + a \bar{m}^\mu + a\bar{a} l^\mu, \qquad \bar{m}^\mu \to \bar{m}^\mu + \bar{a} l^\mu \tag{32}$$

- Null rotations of class II which leave the vector k^μ unchanged:

$$k^\mu \to k^\mu, \qquad m^\mu \to m^\mu + b k^\mu$$
$$l^\mu \to l^\mu + \bar{b} m^\mu + b \bar{m}^\mu + b\bar{b} k^\mu, \qquad \bar{m}^\mu \to \bar{m}^\mu + \bar{b} k^\mu \tag{33}$$

- Null rotations of class III which leave the directions of l^μ and k^μ and the product $l^\mu k_\mu$ unchanged:

$$l^\mu \to \lambda^{-1} l^\mu, \qquad m^\mu \to e^{i\theta} m^\mu$$
$$k^\mu \to \lambda k^\mu, \qquad \bar{m}^\mu \to e^{-i\theta} \bar{m}^\mu \tag{34}$$

1.Every proper orthonormal Lorentz transformation leaves at least one null direction invariant.

The Weyl scalars

As we have seen, the Weyl tensor has, in general, 10 independent components. In the Newman–Penrose formalism those components can be conveniently represented by five complex scalar quantities known as the Weyl scalars, and defined as

$$\Psi_0 := C_{(1)(3)(1)(3)} = C_{\alpha\beta\mu\nu} l^\alpha m^\beta l^\mu m^\nu \tag{35}$$

$$\Psi_1 := C_{(1)(2)(1)(3)} = C_{\alpha\beta\mu\nu} l^\alpha k^\beta l^\mu m^\nu \tag{36}$$

$$\Psi_2 := C_{(1)(3)(4)(2)} = C_{\alpha\beta\mu\nu} l^\alpha m^\beta \bar{m}^\mu k^\nu \tag{37}$$

$$\Psi_3 := C_{(1)(2)(4)(2)} = C_{\alpha\beta\mu\nu} l^\alpha k^\beta \bar{m}^\mu k^\nu \tag{38}$$

$$\Psi_4 := C_{(2)(4)(2)(4)} = C_{\alpha\beta\mu\nu} k^\alpha \bar{m}^\beta k^\mu \bar{m}^\nu \tag{39}$$

Using the definition of the electric and magnetic parts of the Weyl tensor

$$\begin{aligned} \Psi_0 &= Q_{ij} m^i m^j, \qquad & \Psi_1 &= -\frac{1}{\sqrt{2}} Q_{ij} m^i e_r^j \\ \Psi_2 &= \frac{1}{2} Q_{ij} e_r^i e_r^j, \qquad & \Psi_3 &= \frac{1}{\sqrt{2}} Q_{ij} \bar{m}^i e_r^j \\ \Psi_4 &= Q_{ij} \bar{m}^i \bar{m}^j \end{aligned} \tag{40}$$

with $Q_{ij} = E_{ij} - iB_{ij}$, and where $\vec{e}_r$ is the unit radial vector.

The Weyl scalars

We can also invert these relations to express Q_{ij} in terms of the Ψ_a,

$$\Psi_0\bar{m}_i\bar{m}_j + \Psi_4 m_i m_j + \Psi_2 r_i r_j - \sqrt{2}\Psi_1\bar{m}_i r_j + \sqrt{2}\Psi_3 m_i r_j$$

symmetry

$$\Psi_0\bar{m}_i\bar{m}_j + \Psi_4 m_i m_j + \Psi_2 r_i r_j - \sqrt{2}\Psi_1(r_i\bar{m}_j + \bar{m}_i r_j) + \sqrt{2}\Psi_3(r_i m_j + m_i r_j)$$

traceless

$$Q_{ij} = \Psi_0\bar{m}_i\bar{m}_j + \Psi_4 m_i m_j + \Psi_2(2r_i r_j - m_i\bar{m}_j - \bar{m}_i m_j) - \sqrt{2}\Psi_1(r_i\bar{m}_j + \bar{m}_i r_j) + \sqrt{2}\Psi_3(r_i m_j + m_i r_j) \quad (41)$$

The expressions given above for the Ψ_a in terms of the electric and magnetic tensors provide us with a particularly simple way of calculating these scalars in the 3+1 approach: We start from the 3+1 expressions for E_{ij} and B_{ij} and then use these tensors to construct the Ψ_a.

Notice first that the Weyl tensor can be completely specified in terms of the five scalars Ψ_a. On the other hand, the Ψ_a clearly depend on the choice of tetrad. We can then ask if it is possible to make a transformation of the tetrad that will result in one or more of the Ψ_a becoming zero.

For a class I transformation the different Weyl scalars can be shown to transform as,

$$\begin{aligned}
\Psi_0 &\to \Psi_0 \\
\Psi_1 &\to \Psi_1 + \bar{a}\Psi_0 \\
\Psi_2 &\to \Psi_2 + 2\bar{a}\Psi_1 + \bar{a}^2\Psi_0, \\
\Psi_3 &\to \Psi_3 + 3\bar{a}\Psi_2 + 3\bar{a}^2\Psi_1 + \bar{a}^3\Psi_0, \\
\Psi_4 &\to \Psi_4 + 4\bar{a}\Psi_3 + 6\bar{a}^2\Psi_2 + 4\bar{a}^3\Psi_1 + \bar{a}^4\Psi_0,
\end{aligned} \tag{42}$$

Similarly, under a class II transformation we find that

$$\begin{aligned}
\Psi_0 &\to \Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4, \\
\Psi_1 &\to \Psi_1 + 3b\Psi_2 + 3b^2\Psi_3 + b^3\Psi_4, \\
\Psi_2 &\to \Psi_2 + 2b\Psi_3 + b^2\Psi_4, \\
\Psi_3 &\to \Psi_3 + b\Psi_4, \\
\Psi_4 &\to \Psi_4.
\end{aligned} \tag{43}$$

Petrov classification

Let us concentrate now on the class II transformations. It is clear that after such a transformation we can make Ψ_0 vanish as long as we choose the parameter b as one of the roots of the following quartic equation,

$$\Psi_0 + 4b\Psi_1 + 6b^2\Psi_2 + 4b^3\Psi_3 + b^4\Psi_4 = 0 \tag{44}$$

This leads to the Petrov classification that separates different spacetimes into six types according to the number of distinct root of (44),

- Petrov type I: All four roots are distinct: b_1, b_2, b_3, b_4.
  $\{\Psi_s\} \xrightarrow{Trans.\mathrm{II}} \Psi_0 = 0 \xrightarrow{Trans.\mathrm{I}} \Psi_4 = 0 \Rightarrow \{\Psi_1, \Psi_2, \Psi_3\} \neq 0$
- Petrov type II: Two roots coincide: $b_1 = b_2, b_3, b_4$.
  $\{\Psi_s\} \xrightarrow{Trans.\mathrm{II}} \Psi_0 = \Psi_1 = 0 \xrightarrow{Trans.\mathrm{I}} \Psi_4 = 0 \Rightarrow \{\Psi_2, \Psi_3\} \neq 0$
- Petrov type III: Three roots coincide: $b_1 = b_2 = b_3, b_4$.
  $\{\Psi_s\} \xrightarrow{Trans.\mathrm{II}} \Psi_0 = \Psi_1 = \Psi_2 = 0 \xrightarrow{Trans.\mathrm{I}} \Psi_4 = 0 \Rightarrow \{\Psi_3\} \neq 0$
- Petrov type N: All four roots coincide: $b_1 = b_2 = b_3 = b_4$.
  $\{\Psi_s\} \xrightarrow{Trans.\mathrm{II}} \Psi_0 = \Psi_1 = \Psi_2 = \Psi_3 = 0 \Rightarrow \{\Psi_4\} \neq 0$
- Petrov type D: Two pairs of roots coincide: $b_1 = b_2, b_3 = b_4$.
  $\{\Psi_s\} \xrightarrow{Trans.\mathrm{II}} \Psi_0 = \Psi_1 = 0 \xrightarrow{Trans.\mathrm{I}} \Psi_4 = \Psi_3 = 0 \Rightarrow \{\Psi_2\} \neq 0$
- Petrov type O: The Weyl tensor vanishes identically.

Two fundamental complex quantities commonly known as the I and J scalars and defined as,

$$I := \frac{1}{2}\mathcal{C}_{\alpha\beta\mu\nu}\mathcal{C}^{\alpha\beta\mu\nu}, \qquad J := \frac{1}{6}\mathcal{C}_{\alpha\beta\lambda\sigma}\mathcal{C}^{\lambda\sigma}{}_{\mu\nu}\mathcal{C}^{\alpha\beta\mu\nu} \tag{45}$$

where $\mathcal{C}_{\alpha\beta\mu\nu} := \left(C_{\alpha\beta\mu\nu} - iC^{*}_{\alpha\beta\mu\nu}\right)/4.$

The above expressions can be rewritten in terms of the electric and magnetic parts of Weyl tensor as

$$L_B = E^2 - B^2 = 2\Re(I)$$

$$M = E^{\alpha\beta}B_{\alpha\beta} = -\Im(I)$$

$$\begin{aligned} I &= \frac{1}{2}Q_{ab}Q^{ab} \\ &= \frac{1}{2}\left[\left(E_{ab}E^{ab} - B_{ab}B^{ab}\right) - 2iE_{ab}B^{ab}\right], \end{aligned} \tag{46}$$

$$\begin{aligned} J &= -\frac{1}{6}Q_{ij}Q^{i}_{k}Q^{jk} \\ &= -\frac{1}{6}\left[E_{ij}\left(E^{i}_{k}E^{jk} - 3B^{i}_{k}B^{jk}\right) + iB_{ij}\left(B^{i}_{k}B^{jk} - 3E^{i}_{k}E^{jk}\right)\right], \end{aligned} \tag{47}$$

Spacetime Invariants

At the same time, we rewrite I and J in terms of the Weyl scalars Ψ_a:

$$I = 3\Psi_2^2 - 4\Psi_1\Psi_3 + \Psi_0\Psi_4 \tag{48}$$

$$J = \Psi_0\Psi_2\Psi_4 + 2\Psi_1\Psi_2\Psi_3 - \Psi_0\Psi_3^2 - \Psi_1^2\Psi_4 - \Psi_2^3$$

$$= \begin{vmatrix} \Psi_0 & \Psi_1 & \Psi_2 \\ \Psi_1 & \Psi_2 & \Psi_3 \\ \Psi_2 & \Psi_3 & \Psi_4 \end{vmatrix} \tag{49}$$

Invariants can be used to compare a numerical solution to an exact solution, or two numerical solutions corresponding to the same physical system but computed in different gauges.

Spacetime Invariants

For Petrov type D spacetimes like Schwarzschild and Kerr, the only non-zero Weyl scalar is Ψ_2. So we have

$$I = 3\Psi_2^2, \quad J = -\Psi_2^3 \Rightarrow I^3 = 27J^2 \tag{50}$$

so that for a type D spacetime we will always have $I^3 = 27J^2$, regardless of the choice of tetrad. For spacetimes of types III and N we have $I = J = 0$.

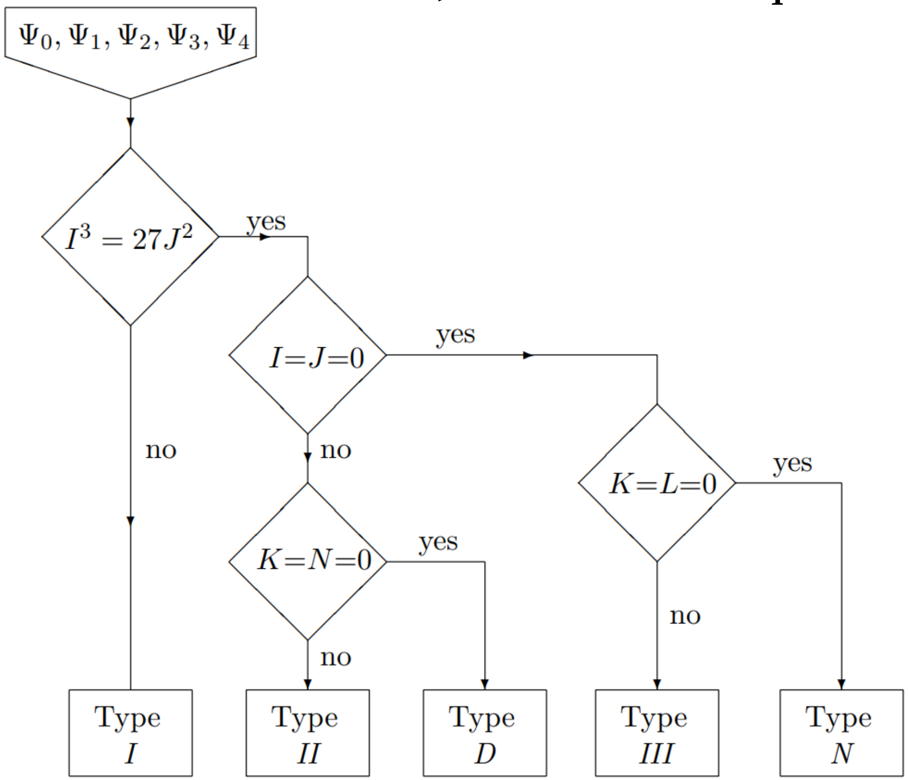
Further scalar invariants are defined as

$$K = \Psi_1 \Psi_4^2 - 3\Psi_4 \Psi_3 \Psi_2 + 2\Psi_3^3 \tag{51}$$

$$L = \Psi_2 \Psi_4 - \Psi_3^2 \tag{52}$$

$$N = 12L^2 - \Psi_4^2 L \tag{53}$$

According to the Petrov classification, we have an equivalent form



Summarize

- Show the advantages of choosing a tetrad basis.
- Construct the relation between the electric and magnetic parts and the Weyl scalars and the invariants.
- Provide the classification of spacetimes.

Thank you for your attention and the questions